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Eigenvalues of a slightly stiff pendulum with a small bob

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SUMMARY

We consider an eigenvalue problem associated with the small vibrations of a slightly stiff pendulum. In this problem, the fourth order differential equation contains two small dimensionless parameters and takes its distinctive character from the simple turning point where the coefficient of the second derivative term vanishes. This turning point always lies outside of the interval of interest. However, a significant feature is that in the parameter range corresponding to small bob mass, one endpoint lies inside the critical layer about the turning point, and hence outer expansions alone are not adequate for formation of a characteristic equation. Approximations to solutions of the governing differential equation are obtained and related by the method of matched asymptotic expansions. All outer expansions are required to be "complete" in the sense of Olver. The ordering of approximations to the characteristic equation is found to depend critically on the relative sizes of the two parameters. We obtain consistent second approximations for the square roots of the eigenvalues.

1. Introduction

In this work, we consider a boundary value problem which arises in connection with the small vibrations of a slightly stiff pendulum. The problem consists of the fourth-order differential equation

$$\varepsilon^3 w^{iv} - \{(1+\delta)-y\} w'' + w' - \lambda^2 w = 0 \qquad (0 \leqq y \leqq 1) \tag{1.1}$$

and the boundary conditions

$$w(0) = w'(0) = w''(1) = \varepsilon^3 w'''(1) - \delta[w'(1) - \lambda^2 w(1)] = 0 \tag{1.2}$$

where $w = w(y)$, λ^2 is the desired eigenvalue, ε and δ are dimensionless parameters, and ε is small, real, and positive. Throughout this work, we shall also assume δ is small, real and positive so that

$$\text{ph}\, \varepsilon = \text{ph}\, \delta = 0 \text{ and } \varepsilon \ll 1, \delta \ll 1 . \tag{1.3}$$

Equation (1.1) has a simple turning point at $y_c = 1 + \delta$ where the coefficient of w'' vanishes. As $\delta > 0$, this turning point always lies *outside* of the interval of interest. However, a significant feature of the boundary value problem is that for δ order ε or smaller, the right-hand endpoint $y = 1$ lies *inside* the critical layer about y_c and a characteristic equation for the eigenvalues cannot be formed using only approximations valid away from the turning point. Indeed, in this case the turning point dominates the problem and the asymptotic behavior of the eigenvalues depends critically on the relative sizes of the two parameters. The eigenvalue also appears in the second boundary condition at $y = 1$.

The reduced equation obtained by formally letting ε tend to zero in Eqn. (1.1) is of second order, and we are thus dealing with a singular perturbation problem in ε. Inner approximations valid at and close to the turning point and outer approximations valid away from the turning point must be obtained separately and then related so that they asymptotically represent the same solutions. We will use the method of matched asymptotic expansions. As usual, the variable y will be assumed complex, and we consider bounded domains in the complex y-plane which contain the real interval [0, 1]. We also require that domains of validity are suitably restricted so that all outer expansions are "complete" in the sense of Olver [8]. Inner and outer

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approximations are matched by means of the asymptotic matching principle in the form stated by Fraenkel [3]. Using these matched expansions, we derive approximations to the characteristic equation and obtain two terms in the asymptotic expansion for λ when δ is order ε or smaller. In analyzing the characteristic equation, to retain consistent ordering different ranges of δ must be considered separately. In all cases, however, we find that the eigenvalues in the present problem are identical at lowest order with the eigenvalues of the limiting problem with $\delta \equiv 0$ where the turning point y_c and endpoint $y=1$ coincide.

The boundary value problem (1.1–2) models the small vibrations of a vertical pendulum consisting of a uniform flexible rod of length L, clamped at one end, with its other end attached to a bob of mass M. If shear deformation and rotary inertia are neglected, linear Euler–Bernoulli theory and an appropriate scaling of physical variables lead to Eqn. (1.1). The second boundary condition at $y=1$ comes from the equation of motion of the bob. If g is the acceleration due to gravity while ρ and EI are the mass per unit length and bending stiffness, respectively, of the rod, the parameters δ and ε correspond to the ratios

$$\varepsilon = \left(\frac{EI}{\rho g L^3}\right)^{\frac{1}{3}} \text{ and } \delta = \frac{M}{\rho L}. \tag{1.4}$$

A slightly stiff rod thus leads to $\varepsilon \ll 1$ while δ measures the ratio of bob mass to rod mass. Physically, λ represents a frequency made dimensionless with respect to the time scale $(L/g)^{\frac{1}{2}}$.

In this paper, we have assumed δ is order ε or smaller so that the turning point is close to the right-hand endpoint. The mathematical limit $\delta \to 0+$ corresponds to the mass of the bob tending to zero. On the other hand, when the mass of the bob is not small compared to the rod mass, δ is order one or larger. This situation has been studied by Handelman and Keller [4]. In their case, the turning point lies relatively far away from the interval of interest. Although the boundary value problem is still a singular perturbation in ε, outer expansions alone are now adequate to form a characteristic equation. Indeed, from a mathematical point of view, the purpose of putting an order one bob mass on a pendulum is to avoid a turning point problem where inner approximations must be used. The situation is quite similar for a rapidly rotating flexible rod. Boyce and Handelman [2] have considered rotating rods with order one tip masses while the turning point problem for the rotating rod alone has been studied by Lakin [5] and Lakin and Ng [6].

To explicitly bring out the turning point nature of the present problem with $\delta \ll 1$, it is convenient to define the Langer variable $\eta(y)$ and a new dependent variable $\phi(\eta)$ by the relations

$$\eta(y) = (1+\delta) - y \text{ and } \phi(\eta) = w(y). \tag{1.5}$$

The turning point y_c now corresponds to $\eta = 0$ while $\eta'(y_c) = -1$ and $\eta''(y_c) = 0$. In terms of the new variables, (1.1) becomes

$$\varepsilon^3 \phi^{iv} - \eta\phi'' - \phi' - \lambda^2 \phi = 0. \tag{1.6}$$

This equation provides a natural starting point for the present theory. We note that δ does not appear explicitly in Eqn. (1.6). This is a major advantage in defining the Langer variable η as we need not at this time make specific assumptions on the relative sizes of δ and ε. The boundary conditions corresponding to Eqn. (1.2) are now

$$\phi(1+\delta) = \phi'(1+\delta) = 0 \tag{1.7}$$

and

$$\phi''(\delta) = \varepsilon^3 \phi'''(\delta) - \delta[\phi'(\delta) + \lambda^2 \phi(\delta)] = 0. \tag{1.8}$$

From Eqn. (1.8), it is clear that the distance between the left-hand endpoint and the turning point is the small parameter δ.

The inner expansions for solutions of Eqn. (1.6) contain functions which exhibit certain symmetries in the complex η-plane. To take full advantage of this fact, we will seek approximations to seven exact solutions of Eqn. (1.6). While the forms of these exact solutions are, of course, unknown, the solutions may be specified by their asymptotic properties. To within a multiplicative constant, we may uniquely define four solutions as follows:

(i) The solution $U_0(\eta)$ is well balanced in bounded domains containing the turning point $\eta=0$.

(ii) The three solutions $V_k(\eta)$ are recessive in the sectors S_k bounded by anti-Stokes lines ($k=1, 2, 3$; see Fig. 1).

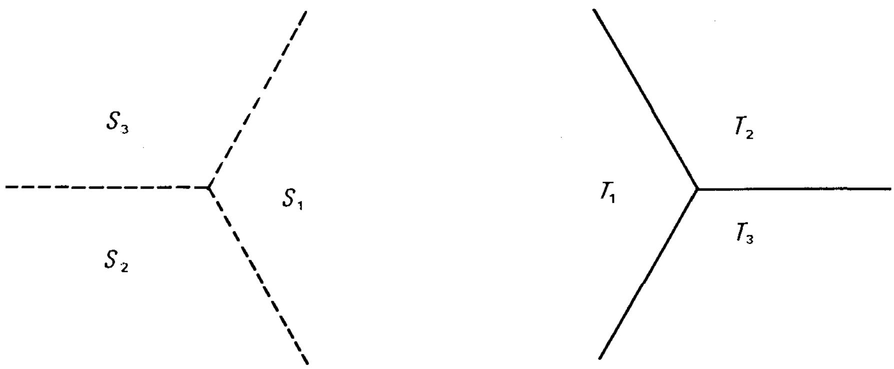

Figure 1. The anti-Stokes lines (left) and the Stokes lines (right) in the η-plane.

To within multiplicative constants and additive multiples of $U_0(\eta)$ we may define three additional solutions by:

(iii) The three solutions $U_k(\eta)$ are purely balanced in the sectors T_k bounded by Stokes lines ($k=1, 2, 3$; see Fig. 1.).

These seven solutions to the fourth order equation must be related through three exact (but unknown) connection formulas. With the exception of $U_0(\eta)$, outer expansions for these solutions will exhibit the Stokes phenomenon. As a byproduct of the matching procedure, we will obtain approximations to the Stokes multipliers which specify the required analytic continuations of outer expansions across Stokes lines in the complex plane.

2. Inner expansions

In this section, we wish to derive inner approximations valid close to the turning point $\eta=0$. Balancing the first, second, and fourth derivative terms in Eqn. (1.6) shows that the critical layer about the turning point will have thickness order ε. Accordingly, we define the stretched variable ξ by

$$\xi=\eta/\varepsilon \tag{2.1}$$

and let

$$\tilde{\phi}(\xi)=\phi(\eta)\,. \tag{2.2}$$

With this scaling Eqn. (1.6) now becomes

$$\tilde{\phi}^{iv}-\xi\tilde{\phi}''-\tilde{\phi}'-\varepsilon\lambda^2\tilde{\phi}=0 \tag{2.3}$$

which suggests seeking expansions of the form

$$\tilde{\phi}=\sum_{n=0}^{\infty}\tilde{\phi}^{(n)}(\xi)\varepsilon^n\,. \tag{2.4}$$

This expansion gives a sequence of differential equations for the $\tilde{\phi}^{(n)}$, and, to write these equations in compact form, we let

$$D=d/d\xi \;\text{ and }\; A=D^2-\xi\,. \tag{2.5}$$

Then

$$DAD\tilde{\phi}^{(0)}=0 \tag{2.6}$$

while

$$DAD\tilde{\phi}^{(n+1)}=\lambda^2\tilde{\phi}^{(n)} \text{ for } n\geqq 0 . \tag{2.7}$$

From these equations, we may obtain partial sums of seven inner expansions which will be denoted by $\tilde{u}_0$, $\tilde{u}_k$, and $\tilde{v}_k$ $(k=1, 2, 3)$. These expansions are defined by the requirement that as $|\xi|\to\infty$ they have the same asymptotic behavior as the exact solutions U_0, U_k and V_k, respectively. Thus, $\tilde{u}_0$ must be well balanced, $\tilde{u}_k$ must be purely balanced as $|\xi|\to\infty$ in T_k, while $\tilde{v}_k$ must be recessive as $|\xi|\to\infty$ in S_k.

Solutions of Eqns. (2.6–7) are expressible in terms of the generalized Airy functions $A_k(\xi, p)$, $B_0(\xi, p)$ and $B_k(\xi; p, q)$ $(k=1, 2, 3)$. These functions were originally developed by Reid [9] in connection with the Orr–Sommerfeld equation and have been adapted by Lakin [5] to study the differential equation of a rapidly rotating flexible rod. In particular, $A_1(z, 0)$ is just the usual Airy function $\mathrm{Ai}(z)$ while $B_k(z; 0, 1)$ satisfies the differential equation

$$(d^2/dz^2-z)\,B_k(z; 0, 1)=1 . \tag{2.8}$$

$B_0(z, p)=B_k(z; p, 0)$ vanishes for $p\leqq 0$ and is a polynomial of degree $p-1$ for $p=1, 2, 3, \dots$. As $|z|\to\infty$, $B_0(z, p)$ is well balanced, $B_k(z; p, q)$ is purely balanced in T_k, while $A_k(z, p)$ is recessive in S_k.

Consider first the inner expansions $\tilde{u}_0$ and $\tilde{u}_k$ of balanced-type. Equation (2.6) and the required asymptotic behavior give that $\tilde{u}_0^{(0)}$ must be a multiple of $B_0(\xi, 1)=1$. Setting $\tilde{u}_0^{(0)}(\xi)=1$ and requiring that $\tilde{u}_0^{(n)}$ contain no multiple of $\tilde{u}_0^{(0)}$ for $n\geqq 1$ now gives

$$\tilde{u}_0(\xi)=1-\varepsilon\lambda^2\xi+\varepsilon^2\,\frac{\lambda^4\xi^2}{4}+O(\varepsilon^3). \tag{2.9}$$

Similarly, $\tilde{u}_k$ must be balanced as $|\xi|\to\infty$ in T_k. Equation (2.6) shows that $\tilde{u}_k^{(0)}$ must thus involve a multiple of $B_k(\xi; 1, 1)$. However, care must be taken as arbitrary multiples of $\tilde{u}_0^{(0)}$ may also be added to $\tilde{u}_k^{(0)}$ without altering the asymptotic behavior. Thus, $\tilde{u}_k^{(0)}$ will be of the form

$$\tilde{u}_k^{(0)}=c_k B_k(\xi; 1, 1)+d_k B_0(\xi, 1)$$

where c_k and d_k are constants independent of ε. For simplicity, we will choose $c_k=1$ and $d_k=0$. However, this choice of d_k does not imply that higher approximations $\tilde{u}_k^{(n)}$ with $n\geqq 1$ will not involve multiples of $B_0(\xi, p)$ with $p>1$. Indeed, for $n\geqq 1$ $\tilde{u}_k^{(n)}$ will involve a polynomial of degree n in ξ as well as a multiple of $\lambda^{2n}B_k(\xi; n+1, 1)$. If we further require that $\tilde{u}_k^{(n)}$ contains no multiple of $\tilde{u}_k^{(0)}$, then Eqn. (2.7) gives

$$\tilde{u}_k(\xi)=B_k(\xi; 1, 1)-\varepsilon\lambda^2[B_k(\xi; 2, 1)+\xi]+\varepsilon^2\,\frac{\lambda^4}{2}\,[B_k(\xi; 3, 1)+\tfrac{3}{4}\xi^2]+O(\varepsilon^3). \tag{2.10}$$

For expansions of recessive type, Eqn. (2.6) and the required asymptotic behavior give that $\tilde{v}_k^{(0)}$ must be a multiple of $A_k(\xi, 1)$. Setting the normalization constant to unity and requiring that $\tilde{v}_k^{(n)}$ contains no multiple of $\tilde{v}_k^{(0)}$ for $n\geqq 1$, we now easily obtain

$$\tilde{v}_k(\xi)=A_k(\xi, 1)-\varepsilon\lambda^2 A_k(\xi, 2)+\varepsilon^2\,\frac{\lambda^4}{2}\,A_k(\xi, 3)+O(\varepsilon^3). \tag{2.11}$$

In general, $\tilde{v}_k^{(n)}$ will be a multiple of $\lambda^{2n}A_k(\xi, n+1)$.

3. Outer expansions

We now wish to derive partial sums of seven outer expansions valid away from the turning point. These expansions will be denoted by $\bar{u}_0$, $\bar{u}_k$ and $\bar{v}_k$ $(k=1, 2, 3)$ and are defined by the conditions that $\bar{u}_0$ is well balanced, $\bar{u}_k$ is purely balanced in T_k, while $\bar{v}_k$ is recessive in S_k. Domains of validity will be suitably restricted so that all expansions are "complete" in the sense of Olver [8].

For the outer expansions $\bar{u}_0$ and $\bar{u}_k$ of balanced-type, we assume a formal expression in powers of ε^3, i.e.

$$\bar{\Phi} = \sum_{n=0}^{\infty} \bar{\Phi}^{(n)}(\eta)\,\varepsilon^{3n}. \tag{3.1}$$

Using Eqn. (3.1) in Eqn. (1.6) and equating powers of ε^3 to zero now gives the sequence of relations

$$\mathscr{L}_2 \bar{\Phi}^{(0)} = 0, \tag{3.2}$$

$$\mathscr{L}_2 \bar{\Phi}^{(n)} = \frac{d^4}{d\eta^4} \bar{\Phi}^{(n-1)} \qquad (n \geqq 1) \tag{3.3}$$

where

$$\mathscr{L}_2 = \eta \frac{d^2}{d\eta^2} + \frac{d}{d\eta} + \lambda^2. \tag{3.4}$$

For applications to the present boundary value problem, we will require only $\bar{u}_0^{(0)}$ and $\bar{u}_k^{(0)}$.

Linearly independent solutions of Eqn. (3.2) involve multiples of the Bessel functions $J_0(2\lambda\eta^{\frac{1}{2}})$ and $Y_0(2\lambda\eta^{\frac{1}{2}})$. The former is regular at $\eta=0$ and may thus be identified with $\bar{u}_0^{(0)}$. Hence

$$\bar{u}_0(\eta) = J_0(2\lambda\eta^{\frac{1}{2}}) + O(\varepsilon^3). \tag{3.5}$$

A second linearly independent solution of Eqn. (3.2) is

$$\bar{u}^{(0)}(\eta) = \frac{\pi}{2} Y_0(2\lambda\eta^{\frac{1}{2}}) \tag{3.6}$$

where the normalization constant $\pi/2$ has been chosen to facilitate matching to the inner expansions. The Bessel function $Y_0(2\lambda\eta^{\frac{1}{2}})$ has a logarithmic branch point at $\eta=0$. To fix the branch in Eqn. (3.6), it is convenient to place a branch cut along the Stokes line dividing T_1 and T_2 and consider ph η in the range

$$-4\pi/3 < \text{ph}\,\eta < 2\pi/3. \tag{3.7}$$

However, a single exact solution of Eqn. (1.6) cannot be asymptotic to $\bar{u}$ for all ph η in this range. In the complete sense, $\bar{u}$ is a valid asymptotic representation for a given exact solution of balanced-type only in a sector of angle $2\pi/3$ bounded by Stokes lines. We thus define the three outer expansions $\bar{u}_k$ by the relations

$$\bar{u}_k(\eta) = \bar{u}(\eta) \text{ for } \eta \in T_k. \tag{3.8}$$

For later use in the central matching problem, we note that $\bar{u}_0^{(0)}$ and $\bar{u}_k^{(0)}$ have the power series expansions

$$\bar{u}_0^{(0)}(\eta) = 1 - \lambda^2\eta + \frac{\lambda^4\eta^2}{4} + O(\eta^3) \tag{3.9}$$

and

$$\bar{u}_k^{(0)}(\eta) = \tfrac{1}{2}\{(\log\eta + \gamma)\bar{u}_0^{(0)}(\eta) + 2\lambda^2\eta + O(\eta^2)\} + (\tfrac{1}{2}\gamma + \log\lambda)\bar{u}_0^{(0)}(\eta) \tag{3.10}$$

where γ is the Euler constant and the coefficient of $\bar{u}_0^{(0)}$ in Eqn. (3.10) has been purposely split into two parts.

Outer expansions $\bar{v}_k$ of recessive type may be obtained using the WKBJ technique. Letting

$$\bar{\Phi}(\eta) = \exp\{\textstyle\int^\eta g(\eta)\,d\eta\} \tag{3.11}$$

and expanding $g(\eta)$ in the form

$$g(\eta) = \varepsilon^{-\frac{3}{2}} g_0(\eta) + g_1(\eta) + \varepsilon^{\frac{3}{2}} g_2(\eta) + \ldots, \tag{3.12}$$

we obtain a sequence of equations for $g_n(\eta)$ beginning with

$$g_0^4 = \eta g_0^2 \tag{3.13}$$

and

$$(4g_0^3 - 2\eta g_0)g_1 = -6g_0^2 g_0' + g_0 + \eta g_0' . \tag{3.14}$$

Equation (3.13) gives $g_0 = 0$ or $g_0 = \pm\eta^{\frac{1}{2}}$. As the trivial solution would lead to an expansion of balanced type, we must have $g_0 = \pm\eta^{\frac{1}{2}}$ and hence

$$g_1(\eta) = -\frac{5}{2}\frac{g_0'}{g_0} + \tfrac{1}{2}g_0^{-2} . \tag{3.15}$$

The transformation (3.11) now gives the approximations

$$\bar{v}_\pm(\eta) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\varepsilon^{\frac{1}{4}}\eta^{-\frac{1}{4}} \exp\{\pm\tfrac{2}{3}\varepsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\} P_\pm(\eta, \varepsilon^{\frac{3}{2}}) \tag{3.16}$$

where $P_\pm(\eta, \varepsilon^{\frac{3}{2}})$ is the Poincaré series

$$P_\pm(\eta, \varepsilon^{\frac{3}{2}}) = 1 + \sum_{n=1}^{\infty} (\pm 1)^n G_n(\eta)\varepsilon^{3n/2} . \tag{3.17}$$

The normalization factor $\frac{1}{2}\pi^{-\frac{1}{2}}\varepsilon^{\frac{1}{4}}$ in Eqn. (3.16) has again been chosen to facilitate matching to the inner expansions, and λ first appears in Eqn. (3.17) at order $\varepsilon^{\frac{3}{2}}$. To fix the branch in these expressions, we again choose ph η in the range (3.7). Beginning with $n=2$, expansions of the $G_n(\eta)$ about $\eta=0$ may contain logarithmic terms.

In the complete sense; a given exact solution of recessive type can be asymptotic to $\bar{v}_+$ or $\bar{v}_-$ only in a sector of angle $4\pi/3$ bounded by Stokes lines. Accordingly, we define the expansions $\bar{v}_1$ and $\bar{v}_2$ by the relations

$$\bar{v}_1 = -\bar{v}_- \quad , \quad \eta \in T_2 \cup T_3 \tag{3.18}$$

and

$$\bar{v}_2 = \quad i\bar{v}_+ \quad , \quad \eta \in T_1 \cup T_3 . \tag{3.19}$$

Care must be taken in the definition of $\bar{v}_3$ as the branch cut lies in the interior of its domain $T_1 \cup T_2$. For $\eta \in T_2$, we take

$$\bar{v}_3 = -i\bar{v}_+, \qquad \eta \in T_2 . \tag{3.20}$$

On crossing the branch cut from T_2 into T_1, $\bar{v}_3$ may change its form but not its value. Hence, taking into account multiples of $2\pi i$ introduced by the logarithmic portions in the $G_n(\eta)$, we must have

$$\bar{v}_3 = \bar{v}_-\{1 + O(\varepsilon^3)\} \text{ for } \eta \in T_1 . \tag{3.21}$$

The expressions for $\bar{u}_k$ and $\bar{v}_k$ are not, of course, valid at the turning point itself. However, in the stated sectors they remain valid asymptotic representations of solutions of Eqn. (1.6) down to $|\eta|$ greater than order ε.

4. The central matching problem and Stokes multipliers

In the central matching problem, one wishes to relate inner and outer expansions so that they represent different asymptotic expansions of the same solution. We will fix normalizations by assuming that the outer expansions of the previous section are asymptotic to the exact solutions of Eqn. (1.6). Hence, when $|\eta|$ is larger than order ε, we have

$$U_0(\eta) \sim \bar{u}_0(\eta), \; U_k(\eta) \sim \bar{u}_k(\eta) \text{ and } V_k(\eta) \sim \bar{v}_k(\eta) . \tag{4.1}$$

Close to the turning point, the exact solutions must now be asymptotic to the combinations

$$\left.\begin{aligned} &U_0(\eta) \sim B_0(\varepsilon)\tilde{u}_0(\xi) , \\ &U_k(\eta) \sim C_k(\varepsilon)\tilde{u}_k(\xi) + B_k(\varepsilon)\tilde{u}_0(\xi) , \\ &V_k(\eta) \sim D_k(\varepsilon)\tilde{v}_k(\xi) \end{aligned}\right\} \tag{4.2}$$

where B_0, B_k, C_k and D_k above, which depend only on ε, are the so-called central matching coefficients. With our normalizations, $\bar{u}_0$ and $\tilde{u}_0$ are simply different representations of $U_0(\eta)$ [compare Eqns. (2.9) and (3.9)] and hence $B_0(\varepsilon)=1$.To determine approximations to the remaining matching coefficients, we will use the asymptotic matching principle with inner and outer expansion operators.

Let $f(\eta, \varepsilon)$ be a given function and let E_p denote the outer expansion operator. Then, $E_p f$ is the outer expansion of f up to and including terms of order ε^p. Similarly, if H_q denotes the inner expansion operator, $H_q f$ is the inner expansion of f up to and including terms of order ε^q. In terms of these operators, the asymptotic matching principle may be written as (see [3])

$$(E_p H_q - H_q E_p) f = 0 \text{ for } n \in \theta \tag{4.3}$$

with θ denoting the overlap region where both inner and outer expansions are valid.

Consider first the coefficients associated with the balanced-type solutions $U_k(\eta)$. To obtain approximations to $B_k(\varepsilon)$ and $C_k(\varepsilon)$ we must apply the matching principle in the sector $\eta \in T_k$ where both $\bar{u}_k$ and $\tilde{u}_k$ (as $|\xi| \to \infty$) are purely balanced. In particular, for $\eta \in T_k \cap \theta$

$$\begin{aligned} H_1 E_0 U_k &= H_1 \bar{u}_k^{(0)}(\varepsilon\xi) \\ &= \tfrac{1}{2}\{(\log\xi+\gamma)(1-\varepsilon\lambda^2\xi)+2\varepsilon\xi\lambda^2\}+\tfrac{1}{2}\{(\log\varepsilon+\log\lambda^2+\gamma)(1-\varepsilon\lambda^2\xi)\} \end{aligned} \tag{4.4}$$

while

$$E_0 H_1 U_k = E_0 H_1 \{C_k(\varepsilon)\tilde{u}_k(\xi) + B_k(\varepsilon)\tilde{u}_0(\xi)\}\,. \tag{4.5}$$

These equations suggest that the matching coefficients have expansions with respect to the asymptotic sequence $\log\varepsilon$, 1, $\varepsilon\log\varepsilon$, ε, $\varepsilon^2\log^2\varepsilon$, $\varepsilon^2\log\varepsilon$, ε^2, … . Matching in $T_k \cap \theta$ with $p=3$ and $q=4$, we find that

$$B_k(\varepsilon) = \tfrac{1}{2}\{\log\varepsilon + (\gamma + 2\log\lambda) + O(\varepsilon^3)\} \tag{4.6}$$

while

$$C_k(\varepsilon) = -\tfrac{1}{2} + O(\varepsilon^3)\,. \tag{4.7}$$

Because of the rapidly varying exponential factors in Eqn. (3.16), the expansion operator E_p cannot be applied directly to solutions V_k of dominant-recessive type. Rather, we must use the modified outer expansion operator

$$E_p^{\pm} = G^{\pm} E_p G^{\mp} \tag{4.8}$$

with

$$\begin{aligned} G^{\pm} &= \tfrac{1}{2}\pi^{-\frac{1}{2}}\varepsilon^{\frac{3}{4}}\eta^{-\frac{3}{4}}\exp\{\pm\tfrac{2}{3}\varepsilon^{-\frac{3}{2}}\eta^{\frac{3}{2}}\} \\ &= \tfrac{1}{2}\pi^{-\frac{1}{2}}\xi^{-\frac{3}{4}}\exp\{+\tfrac{2}{3}\xi^{\frac{3}{2}}\} \end{aligned} \tag{4.9}$$

and the matching principle

$$(E_p^{\pm} H_q - H_q E_p^{\pm}) f = 0 \text{ for } \eta \in \theta\,. \tag{4.10}$$

To derive approximations to $D_k(\varepsilon)$, the matching must be done in the sector $\eta \in S_k$ where both $\bar{v}_k$ and $\tilde{v}_k$ (as $|\xi| \to \infty$) are purely recessive. A short calculation gives

$$D_k(\varepsilon) = 1 + O(\varepsilon^3 \log\varepsilon)\,. \tag{4.11}$$

Outer expansions for the six exact solutions U_k and $V_k (k=1, 2, 3)$ exhibit the Stokes phenomenon and, in the complete sense, are valid asymptotic representations only in certain restricted sectors of the complex η-plane which have Stokes lines as boundaries. The continuation of complete outer expansions across bounding Stokes lines is known as the lateral connection problem. Using results from the central matching problem and exact connection formulas for the generalized Airy functions in the inner expansions, we may now obtain the required analytic continuations directly. For simplicity, only the two representative solutions U_3 and V_2 will be considered in detail.

Let $\bar{u}_0$, $\bar{u}_k$ and $\bar{v}_k$ denote the expressions in section three. Then, in the complete sense, U_3 is asymptotic to $\bar{u}_3$ for $\eta \in T_3$. On crossing the Stokes line into T_2, the outer expansion of U_3 must discontinuously pick up a multiple of the recessive outer expansion $\bar{v}_1$. Similarly, in T_1 the outer expansion of U_3 must contain multiples of both $\bar{v}_2$ and $\bar{u}_0$. Hence, if s_0, s_1 and s_2 are

Stokes multipliers which characterize the continuations, we have

$$U_3 \sim \begin{cases} \bar{u}_3(\eta) & (\eta \in T_3) \\ \bar{u}_1(\eta)+s_0\bar{u}_0(\eta)+s_1\bar{v}_2(\eta) & (\eta \in T_1) \\ \bar{u}_2(\eta)+s_2\bar{v}_1(\eta) & (\eta \in T_2) \end{cases} \tag{4.12}$$

The exact connection formulas

$$B_3(\xi\,;p,1)=B_1(\xi\,;p,1)+2\pi i[A_2(\xi,p)+B_0(\xi,p)]\,,$$
$$B_3(\xi\,;p,1)=B_2(\xi\,;p,1)-2\pi i A_1(\xi,p)$$

and the approximations to $B_k(\varepsilon)$ and $C_k(\varepsilon)$ now give immediately that

$$s_0=s_1=-s_2=-\pi i[1+O(\varepsilon^3)] \tag{4.13}$$

Similarly, for V_2 we must have

$$V_2 \sim \begin{cases} \bar{v}_2(\eta) & (\eta \in T_1 \cup T_3) \\ -\bar{v}_3(\eta)+s_3\bar{v}_1(\eta)+s_4\bar{u}_0(\eta) & (\eta \in T_2) \end{cases} \tag{4.14}$$

The approximation to $D_2(\varepsilon)$ and the exact relations

$$\sum_{k=1}^{3} A_k(\xi,p)=-B_0(\xi,p)$$

now give for the Stokes multipliers associated with this solution

$$s_3=s_4=-1+O(\varepsilon^3\log\varepsilon)\,. \tag{4.15}$$

For use in the boundary value problem, we will need outer expansion for both $U_3(\eta)$ and $V_2(\eta)$ directly on the Stokes line ph $\eta=0$. Lakin and Ng [6] have shown that the correct form for a complete outer expansion on a Stokes line itself is the mean of the complete expansions valid on either side of the line. Hence, for ph $\eta=0$

$$\left.\begin{aligned} &U_3(\eta)\sim\bar{u}_3(\eta)+\frac{s_2}{2}\bar{v}_1(\eta) \\ &\text{while} \\ &V_2(\eta)\sim\bar{v}_2(\eta)+\frac{s_3}{2}\bar{v}_1(\eta)+\frac{s_4}{2}\bar{u}_0(\eta)\,. \end{aligned}\right\} \tag{4.16}$$

We further note that on the Stokes line ph $\eta=0$, $U_0(\eta)\sim\bar{u}_0(\eta)$ is well balanced while $V_1(\eta)\sim\bar{v}_1(\eta)$ is maximally recessive.

5. The boundary value problem

We now wish to use the matched expansions derived in the previous sections to study the boundary value problem Eqns. (1.6–8). A fundamental set of linearly independent solutions of Eqn. (1.6) consists of U_0, one of the three solutions U_k, and two of the three solutions V_k. We will choose the set $\{U_0, U_3, V_1, V_2\}$ which is "numerically satisfactory" in the sense of Miller [7]. Conditions (1.7–8) now lead to a characteristic equation for the eigenvalues involving a four-by-four determinant. Let $\mathscr{B}_3$ denote the operator

$$\mathscr{B}_3=\varepsilon^3\frac{d^3}{d\eta^3}-\delta\left(\frac{d}{d\eta}+\lambda^2\right)=\frac{d^3}{d\xi^3}-\delta\left(\varepsilon^{-1}\frac{d}{d\xi}+\lambda^2\right). \tag{5.1}$$

Further, let

$$W(X,Y)=X(\eta_2)Y'(\eta_2)-X'(\eta_2)Y(\eta_2) \tag{5.2}$$

denote the Wronskianof X and Y at $\eta_2 = 1+\delta$ and

$$B(X, Y) = \mathscr{B}_3 X(\eta_1) \frac{d^2 Y}{d\eta^2}(\eta_1) - \frac{d^2 X}{d\eta^2}(\eta_1) \mathscr{B}_3 Y(\eta_1)$$

with $\eta_1 = \delta$. Then, on expanding the determinant, we obtain the exact characteristic equation

$$\begin{aligned} &B(U_0, U_3) W(V_1, V_2) - B(U_0, V_1) W(U_3, V_2) + B(U_0, V_2) W(U_3, V_1) \\ &\quad + B(U_3, V_1) W(U_0, V_2) - B(U_3, V_2) W(U_0, V_1) + B(V_1, V_2) W(U_0, U_3) = 0 \end{aligned} \tag{5.4}$$

As the boundary point $\eta_2 = 1+\delta$ is always far away from the turning point $\eta = 0$, the Wronskians in Eqn. (5.4) may be consistently approximated using complete outer expansions. On the other hand, with our assumption that δ is at most order ε, the boundary point $\eta_1 = \delta$ lies inside the critical layer about $\eta = 0$. As a result, the matched inner expansions must be used to consistently approximate the quantities $B(X, Y)$.

For example,

$$W(U_3, V_1) = W(\bar{u}_3, \bar{v}_1) \tag{5.5}$$

but

$$B(U_3, V_1) = D_1(\varepsilon)\{C_3(\varepsilon) B(\tilde{u}_3, \tilde{v}_1) + B_3(\varepsilon) B(\tilde{u}_0, \tilde{v}_1)\}\,. \tag{5.6}$$

Using the appropriate expansions in Eqn. (5.4) now leads to an approximate characteristic equation which contains three distinct types of terms and may be written in the form

$$\mathscr{D} + \mathscr{B} + \mathscr{R} = 0\,. \tag{5.7}$$

Terms in $\mathscr{D}$ involve a multiple of either $W(\bar{u}_0, \bar{v}_2)$ or $W(\bar{u}_3, \bar{v}_2)$ and are thus dominant as they contain the exponential factor

$$E(\eta_2) = \exp\{+\tfrac{2}{3}\varepsilon^{-\frac{3}{2}}\eta_2^{\frac{3}{2}}\}. \tag{5.8}$$

In particular,

$$\mathscr{D} = D_1(\varepsilon) W(\bar{u}_0, \bar{v}_2)[C_3(\varepsilon) B(\tilde{u}_3, \tilde{v}_1) + B_3(\varepsilon) B(\tilde{u}_0, \tilde{v}_1)] - D_1(\varepsilon) W(\bar{u}_3, \bar{v}_2) B(\tilde{u}_0, \tilde{v}_1)\,. \tag{5.9}$$

On the other hand, terms in $\mathscr{B}$ contain no exponential factors and are balanced while terms in $\mathscr{R}$ contain the exponential factor $E^{-1}(\eta_2)$ and are recessive. Because of these exponentials, for $\varepsilon \ll 1$ we have

$$\mathscr{D} \gg \mathscr{B} \gg \mathscr{R}\,. \tag{5.10}$$

It is particularly convenient to explicitly take account of the rapidly varying exponentials and re-write Eqn. (5.7) as

$$\Delta_1(\varepsilon, \delta) E(\eta_2) + \Delta_2(\varepsilon, \delta) + \Delta_3(\varepsilon, \delta) E^{-1}(\eta_2) = 0\,. \tag{5.11}$$

This form shows that the limiting behavior of the eigenvalues comes from the leading term in the approximate equation

$$\Delta_1(\varepsilon, \delta) = 0\,. \tag{5.12}$$

Indeed, Eqn. (5.12) remains a valid approximation to the full characteristic equation over a wide range of ε. This is due to the fact that η_2 lies on the Stokes line ph $\eta = 0$ in the complex η-plane and hence $E(\eta_2)$ and $E^{-1}(\eta_2)$ are maximally dominant and recessive, respectively. As a result, both $\mathscr{B}$ and $\mathscr{R}$ will remain small compared to error terms in $\mathscr{D}$, even for relatively large values of ε ($\varepsilon = 0.1$, say). Equation (5.12) may thus be used to obtain consistent second approximation: to $\lambda(\varepsilon, \delta)$.

We now wish to derive approximations to $\Delta_1(\varepsilon, \delta)$. From Eqns. (2.10–11) and (5.1), the quantity $B(\tilde{u}_3, \tilde{v}_1)$ is order ε^{-2}. By contrast, both $\varepsilon^{-2}(d^2/d\xi^2)\tilde{u}_0(\xi_1)$ and $\delta^{-2}\mathscr{B}_3\tilde{u}_0(\xi_1)$ with $\xi_1=\eta_1/\varepsilon$ are order one. Further, the leading term in $\mathscr{B}_3\tilde{v}_1(\xi_1)$ is $-\varepsilon\lambda^2 A_1(\delta/\varepsilon, -1)$. Hence

$$B(\tilde{u}_0, \tilde{v}_1)=O(\varepsilon, \varepsilon^{-2}\delta^2)\,.$$

For second approximations to $\lambda(\varepsilon, \delta)$, we need thus consider only the product $W(\bar{u}_0, \bar{v}_2)B(\tilde{u}_3, \tilde{v}_1)$ in Eqn. (5.12).

The expansion for the Wronskian $W(\bar{u}_0, \bar{v}_2)$ involves $E(\eta_2)$ times a Poincaré series in powers of $\varepsilon^{\frac{3}{2}}$ with coefficients depending on δ. As δ is small, the expansion of these coefficients in powers of δ is a straightforward process. Similarly, the expansion for $B(\tilde{u}_3, \tilde{v}_1)$ is a series in powers of ε involving Airy functions evaluated at δ/ε. If δ is less than order ε, these Airy functions may in turn be re-expanded relative to the turning point as series in δ/ε. The special case $\delta=O(\varepsilon)$ is treated at the end of this section.

Up to this point, we have only assumed δ does not exceed order ε. It has not been necessary to make explicit assumptions about the relative sizes of δ and ε, nor have we assumed a particular expansion for $\lambda(\varepsilon, \delta)$. However, to consistently order approximations to $W(\bar{u}_0, \bar{v}_2)B(\tilde{u}_3, \tilde{v}_1)$, we must now treat different ranges of δ separately. In what follows, we will use Hardy's notation "$\prec$" and "$\succ$" for relative orders of magnitude. For example "$f\prec g$" may be read "f is of small order than g".

Range 1: $\delta\prec\varepsilon^{\frac{3}{2}}$.

In this case, terms of order δ and $(\delta/\varepsilon)^2$ in $W(\bar{u}_0, \bar{v}_2)$ and $B(\tilde{u}_3, \tilde{v}_1)$ may be neglected relative to terms of order $\varepsilon^{\frac{3}{2}}$ and ε, respectively. The consistently ordered approximation to $\Delta_1(\varepsilon, \delta)=0$ is now of the form

$$J_0(2\lambda)\left\{1-\lambda^2\varepsilon\frac{Ai(0)}{Ai'(0)}+O(\delta/\varepsilon)^2\right\}$$
$$+\varepsilon^{\frac{3}{2}}\{J_0(2\lambda)[G_1(1)-\tfrac{3}{4}]+\lambda J_1(2\lambda)\}+O(\varepsilon^2\log\varepsilon)=0 \tag{5.13}$$

where $J_0(z)$, $J_1(z)$ are Bessel functions, $Ai(z)$ is the usual Airy function, and Ai(0), Ai′(0) are non-zero constants. Equation (5.13) suggests an expansion for λ of the form

$$\lambda=\lambda^{(0)}+\varepsilon^{\frac{3}{2}}\lambda^{(1)}+\dots\,. \tag{5.14}$$

We now find that $\lambda^{(0)}$ is determined by the relation $J_0(2\lambda^{(0)})=0$, i.e.

$$\lambda_n^{(0)}=\tfrac{1}{2}j_{0,n} \tag{5.15}$$

where $j_{0,n}$ is the n-th positive zero of the Bessel function J_0. Further, $\lambda^{(1)}=\frac{1}{2}\lambda^{(0)}$ and hence

$$\lambda_n=\tfrac{1}{2}j_{0,n}\left\{1+\frac{\varepsilon^{\frac{3}{2}}}{2}+\dots\right\}. \tag{5.16}$$

To this order, Eqn. (5.16) is identical with the corresponding result for the limiting problem of a bob-less pendulum. At the earliest, δ enters the expansion for λ at third order.

Range 2: $\delta=O(\varepsilon^{\frac{3}{2}})$.

This range of δ is a transition case. Consistent ordering of approximations to $\Delta_1(\varepsilon, \delta)$ is now a delicate matter as many terms in the product $W(\bar{u}_0, \bar{v}_2)B(\tilde{u}_3, \tilde{v}_1)$ are of the same order and, in the strict sense, must be included. In particular, $(\delta/\varepsilon)^2$ is order ε while δ, $(\delta/\varepsilon)^3$, and $\varepsilon(\delta/\varepsilon)$ are all order $\varepsilon^{\frac{3}{2}}$. Fortunately, the first approximation $\lambda^{(0)}$ again comes from the vanishing of $J_0(2\lambda^{(0)})$. Hence, it is sufficient to write the approximation to $\Delta_1(\varepsilon, \delta)=0$ in this case as

$$J_0(2\lambda)Q(\delta, \varepsilon)+\lambda J_1(2\lambda)[\varepsilon^{\frac{3}{2}}-\delta+O(\varepsilon^2)]=0 \tag{5.17}$$

where

$$Q(\delta, \varepsilon) = 1 + O\left(\frac{\delta}{\varepsilon}\right). \tag{5.18}$$

If, in addition, we now let

$$\delta = \sigma\,\varepsilon^{\frac{3}{2}} \tag{5.19}$$

where σ is an order one constant independent of ε, expanding λ as in Eqn. (5.14) gives

$$\left.\begin{aligned} \lambda_n^{(0)} &= \tfrac{1}{2} j_{0,n}\,, \\ \lambda_n^{(0)} &= \frac{1-\sigma}{2}\,\lambda_n^{(0)}\,. \end{aligned}\right\} \tag{5.20}$$

We thus have

$$\lambda_n = \tfrac{1}{2} j_{0,n} \left\{ 1 + \frac{1}{2}(\varepsilon^{\frac{3}{2}} - \delta) + \ldots \right\}. \tag{5.21}$$

The slight rod stiffness and small bob mass now influence the eigenvalues simultaneously and produce a second order deviation from the limiting behavior.

Range 3: $\varepsilon^{\frac{3}{2}} \prec \delta \prec \varepsilon$.

Terms of order $\varepsilon^{\frac{3}{2}}$ may now be consistently neglected relative to terms of order δ, and we obtain as an approximation to $\Delta_1(\varepsilon, \delta) = 0$ the equation

$$J_0(2\lambda)\left\{1 + O\left(\frac{\delta}{\varepsilon}\right)\right\} - \lambda\delta J_1(2\lambda)\left\{1 + O\left(\frac{\varepsilon^{\frac{3}{2}}}{\delta}\right)\right\} = 0\,. \tag{5.22}$$

This suggests an expansion for λ of the form

$$\lambda = \lambda^{(0)} + \delta\lambda^{(1)} + \ldots \tag{5.23}$$

which gives

$$\lambda_n = \tfrac{1}{2} j_{0,n} \left\{ 1 - \frac{\delta}{2} + \ldots \right\}. \tag{5.24}$$

The parameter ε now does not enter the expansion for λ until higher order.

Range 4: $\delta = O(\varepsilon)$.

For convenience, we let $\delta = \sigma\varepsilon$ where σ is again a non-zero constant independent of ε. In this range, powers of (δ/ε) are all order one so, for the present purposes, the generalized Airy functions in $B(\tilde{u}_3, \tilde{v}_1)$ cannot be expanded relative to the turning point. Leaving these functions evaluated at $\delta/\varepsilon = \sigma$, we obtain the approximation to $\Delta_1(\varepsilon, \delta) = 0$

$$J_0(2\lambda)\left\{1 - \varepsilon\left[\lambda^2\,\beta(\sigma) + \frac{\sigma}{4}\right]\right\} - \varepsilon\lambda\sigma J_1(2\lambda) + O(\varepsilon^{\frac{3}{2}}) = 0 \tag{5.25}$$

where

$$\beta(\sigma) = [\mathrm{Ai}(\sigma) - \sigma\mathrm{Ai}'(\sigma)]/\mathrm{Ai}'(\sigma)$$

is a non-zero constant. Expanding λ in the form $\lambda = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \ldots$, we now recover the result, Eqn. (5.24).

6. Discussion

The distinguishing feature of this boundary value problem is the singular perturbation in the small parameter ε coupled with the presence of a second parameter δ which measures the distance between an endpoint and a simple turning point of the governing differential equation. Even though it lies outside of the interval of interest, for $\delta \preccurlyeq \varepsilon$ this turning point cannot be ignored, and inner expansions as well as outer expansions must be used in the boundary value problem. We have derived partial sums of the required matched inner and outer expansions without making explicit assumptions on the size of δ. However, the correct ordering in approximations to the characteristic equation (and hence the relevant expansion for $\lambda(\varepsilon, \delta)$) is critically dependent on the relative sizes of the two parameters.

Equations (5.16), (5.21), and (5.24) show that over the entire parameter range $\delta \preccurlyeq \varepsilon$ the expansions for λ agree at lowest order. However, there is a smooth transition in λ at second order from $\delta \prec \varepsilon^{\frac{3}{2}}$ through $\delta = O(\varepsilon)$ with $\delta = O(\varepsilon^{\frac{3}{2}})$ representing a transition range. Indeed, if we define the step functions $\alpha_1(\delta)$ and $\alpha_2(\delta)$ by

$$\alpha_1 = \begin{cases} 1 & \delta \preccurlyeq \varepsilon^{\frac{3}{2}} \\ 0 & \varepsilon^{\frac{3}{2}} \prec \delta \preccurlyeq \varepsilon \end{cases} \tag{6.1}$$

and

$$\alpha_2 = \begin{cases} 0 & \delta \prec \varepsilon^{\frac{3}{2}} \\ 1 & \varepsilon^{\frac{3}{2}} \preccurlyeq \delta \preccurlyeq \varepsilon\,, \end{cases} \tag{6.2}$$

the, the three expressions for $\lambda(\varepsilon, \delta)$ may be condensed into the single consistent second approximation

$$\lambda_n = \tfrac{1}{2} j_{0,n} \left\{ 1 + \frac{1}{2}(\alpha_1 \varepsilon^{\frac{3}{2}} - \alpha_2 \delta) + \dots \right\}. \tag{6.3}$$

Hence, for $\delta \prec \varepsilon^{\frac{3}{2}}$, at earliest δ contributes to λ at third order. On the other hand, when $\varepsilon^{\frac{3}{2}} \preccurlyeq \delta \preccurlyeq \varepsilon$, δ contributes to the expansion of λ at second order, and tends to decrease λ. In terms of the physical scales, the parameter range $\varepsilon^{\frac{3}{2}} \preccurlyeq \delta \preccurlyeq \varepsilon$ corresponds to

$$\left(\frac{\rho EI}{gL}\right)^{\frac{1}{2}} \preccurlyeq M \preccurlyeq \left(\frac{\rho^2 EI}{g}\right)^{\frac{1}{3}}. \tag{6.4}$$

Thus for small bobs in this mass range, increasing the bob mass will produce a second order decrease in the natural frequencies of vibration.

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